Note on minimally k-rainbow connected graphs*

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Abstract

An edge-colored graph G, where adjacent edges may have the same color, is *rainbow connected* if every two vertices of G are connected by a path whose edge has distinct colors. A graph G is *k-rainbow connected* if one can use k colors to make G rainbow connected. For integers n and d let $t(n,d)$ denote the minimum size (number of edges) in k-rainbow connected graphs of order n. Schiermeyer got some exact values and upper bounds for $t(n,d)$. However, he did not get a lower bound of $t(n,d)$ for $3 \leq d < \lceil \frac{n}{2} \rceil$. In this paper, we improve his lower bound of $t(n,2)$, and get a lower bound of $t(n,d)$ for $3 \leq d < \lceil \frac{n}{2} \rceil$.

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1 Introduction

A communication network consists of nodes and links connecting them. In order to prevent hackers, one can set a password in each link. To facilitate the management, one can require that the number of passwords is small enough such that every two nodes can exchange information by a sequence of links which have different passwords. This problem can be modeled by a graph and studied by means of rainbow connection.

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All graphs in this paper are undirected, finite and simple. We refer to book [1] for notation and terminology not described here. A path in an edge-colored graph G, where adjacent edges may have the same color, is a *rainbow path* if no pair of edges are colored the same. An edge-coloring of G with k colors is a *k-rainbow connected coloring* if every two distinct vertices of G are connected by a rainbow path. A graph G is *k-rainbow connected* if G has a k-rainbow connected coloring. Note that Schiermeyer used the term *rainbow k-connected in [12].* However we think that it is better to use the term *k-rainbow connected* since this will distinguish it from the term *rainbow k-connectivity*, which means that there are many rainbow paths between every pair of vertices, see [6]. The rainbow connection number $rc(G)$ of G is the minimum integer k such that G has a k-rainbow connected coloring. It is easy to see that $rc(G) \geq diam(G)$ for any connected graph G, where $diam(G)$ is the diameter of G.

The rainbow connection number was introduced by Chartrand et al. in [5]. Bounds on the rainbow connection numbers of graphs have been studies in terms of other graph parameters, such as radius, dominating number, minimum degree, connectivity, etc., see [2, 4, 5, 8, 9, 10, 11]. In [3], Chakraborty et al. investigated the hardness and algorithms for the rainbow connection number, and showed that given a graph G, deciding if $rc(G) = 2$ is NP-complete. In particular, computing $rc(G)$ is NP-hard.

For integers n and d let $t(n, d)$ denote the minimum size (number of edges) in k-rainbow connected graphs of order n. Because a network which satisfies our requirements and has as less links as possible can cut costs, reduce the construction period and simplify later maintenance, the study of this parameter is significant. Schiermeyer [12] mainly investigated the upper bound of $t(n, d)$ and showed the following results.

Theorem 1. (Schiermeyer[12])

(i) $t(n, 1) = \binom{n}{2}$.

(ii) $t(n, 2) \leq (n + 1)\lfloor \log_2 n\rfloor - 2^{\lfloor \log_2 n\rfloor} - 2$.

(iii) $t(n, 3)) \leq 2n - 5$.

(iv) For $4 \leq d < \frac{n-1}{2}$, $t(n, d) \leq n - 1 + \left\lceil \frac{n-2}{d-2} \right\rceil$.

(v) For $\frac{n}{2} \leq d \leq n - 2$, $t(n, d) = n$.

(vi) $t(n, n - 1) = n - 1$.

In [12], Schiermeyer also got a lower bound of $t(n, 2)$ by an indirect method, and showed that $t(n, 2) \geq n \log_2 n - 4n \log_2 \log_2 n - 5n$ for sufficiently large n. Nevertheless, he did not get a lower bound of $t(n, d)$ for $3 \leq d < \lceil \frac{n}{2} \rceil$. In this paper, we use a different method to improve his lower bound of $t(n, 2)$, and moreover we get a lower bound of $t(n, d)$ for $3 \leq d < \lceil \frac{n}{2} \rceil$.

2 Main results

Let G be a 2-rainbow connected graph of order n with maximum degree $\Delta(G)$. Pick a vertex $u \in V(G)$. Since $d(u) \leq \Delta(G)$, there exist at most $\Delta(G)$ vertices adjacent to u, and at most $\Delta(G)(\Delta(G)-1)$ vertices at distance 2 from u. Since $diam(G) \leq rc(G) \leq 2$, we derive $n \leq 1 + \Delta(G) + \Delta(G)(\Delta(G)-1)$. Thus, $\Delta(G) \geq \sqrt{n-1}$. Since $\Delta(G)$ is an integer, we get

$$\Delta(G) \geq \left\lceil \sqrt{n-1} \right\rceil . \tag{1}$$

Next, we investigate the lower bound of $e_2(n)$.

Proposition 1. *For sufficiently large n, $t(n,2) \geq n\log_2 n - 4n\log_2\log_2 n - 2n$.*

Proof. Let G be a graph with diameter 2 and c be a 2-rainbow connected coloring of G with colors blue and red. Set $k = \lfloor \log_2 n \rfloor^2 - 1$ and denote by S the set of vertices with degrees less than k. Assume that $S = \{u_1, u_2, \ldots, u_s\}$ and $T = V(G)\backslash S = \{u_{s+1}, u_{s+2}, \ldots, u_{s+t}\}$, where $s + t = n$. For sufficiently large n, $k = \lfloor \log_2 n \rfloor^2 - 1$ $\leq \left\lceil \sqrt{n-1} \right\rceil \leq \Delta(G)$. By (1) we know that T is nonempty. If $t = |T| \geq \frac{2n}{\log_2 n}$, then

$$\begin{aligned} e(G) \geq \frac{1}{2}\sum_{v\in T} d_G(v) &\geq \frac{2n}{2\log_2 n}\left(\lfloor \log_2 n \rfloor^2 - 1\right) \\ &\geq \frac{n}{\log_2 n}\left((\log_2 n - 1)^2 - 1\right) \\ &= n\log_2 n - 2n, \end{aligned}$$

and we are done.

Suppose $t < \frac{2n}{\log_2 n}$, that is, $s > n - \frac{2n}{\log_2 n}$. It is sufficient to show that $e(S,T) \geq n\log_2 n - 4n\log_2\log_2 n - 2n$.

For every u_i, $1 \leq i \leq s$, we define a vector as follows:

$$\alpha(u_i) = (b_{i,1}, b_{i,2}, \ldots, b_{i,t}),$$

where

$$b_{i,j} = \begin{cases} 1, & if\ c(u_i u_{s+j})\ is\ red; \\ -1, & if\ c(u_i u_{s+j})\ is\ blue; \\ 0, & if\ u_i\ and\ u_{s+j}\ is\ nonadjacent. \end{cases}$$

Suppose $|N(u_i) \cap T| = a_i$, where $1 \leq i \leq s$. Then $e(S,T) = \sum_{i=1}^{s} a_i$, where $e(S,T)$ denotes the number of edges between S and T. We now estimate the value

of $e(S,T)$. For each $\alpha(u_i)$, we define a set B_i as follows: $B_i = \{$vectors obtained from $\alpha(u_i)$ by replace "0" of $\alpha(u_i)$ by "1" or "-1"$\}$. Because $|N(u_i) \cap T| = a_i$, we have $|B_i| = 2^{t-a_i}$ for each i, where $1 \le i \le s$. Set $B = \bigcup_{i=1}^{s} B_i$. Then B is a multiset of t-dimensional vectors with elements 1 and -1. For each $\alpha \in B$, n_α denotes the number of α's in B. We have the following claim.

Claim 1. For each $\alpha \in B$, $n_\alpha \le k^2 + 1$.

Proof of Claim 1. If Claim 1 is not true, that is, there exists a vector α, without loss of generality, say $\alpha = (b_1, b_2, \ldots, b_t)$, such that $n_\alpha \ge k^2 + 2$. Clearly, it is not possible that there exists some B_i such that B_i contains two α's. Thus, there exist $k^2 + 2$ integers, without loss of generality, say $1, 2, \ldots, k^2 + 2$, such that $\alpha \in B_i$, where $1 \le i \le k^2 + 2$. we next show that for each i, where $2 \le i \le k^2 + 2$, the distance between u_1 and u_i in $G[S]$ is at most 2. In fact, $c(u_1 u_{s+j}) = b_j = c(u_i u_{s+j})$ follows from the definition of B_1 and B_i. Thus there exists no rainbow path between u_1 and u_i through a vertex of T. Hence, there must exist a rainbow path between u_1 and u_i with length at most 2 in $G[S]$. On the other hand, since $\Delta(G[S]) \le k$, the number of vertices at distance 2 from u_1 is at most $k^2 + 1$, which is a contradiction, and the claim is thus true.

By Claim 1 we know

$$\sum_{i=1}^{s} |B_i| \le (k^2 + 1) 2^t,$$

Since $|B_i| = 2^{t-a_i}$ for each i, where $1 \le i \le s$,

$$\sum_{i=1}^{s} 2^{-a_i} \le (k^2 + 1).$$

By the inequality between the geometrical and arithmetical means, we have

$$\sqrt[s]{2^{-e(S,T)}} = \sqrt[s]{2^{-\sum_i^s a_i}} \le \frac{1}{s} \sum_{i=1}^{s} 2^{-a_i} \le \frac{k^2+1}{s}.$$

Using the log function on both sides, we get

$$e(S,T) \ge s \log_2 s - s \log_2 (k^2 + 1). \tag{2}$$

Note that $k^2 + 1 = (\lfloor \log_2 n \rfloor^2 - 1)^2 + 1 \le ((\log_2 n)^2 - 1)^2 + 1 \le (\log_2 n)^4 - 2(\log_2 n)^2 + 2 \le (\log_2 n)^4$. We have

$$e(S,T) \ge s \log_2 s - 4s \log_2 \log_2 n. \tag{3}$$

Since $e(S,T)$ is monotonically increasing in s and $s > n - \frac{2n}{\log_2 n}$, we have

$$e(S,T) \ \ge \ \left(n - \frac{2n}{\log_2 n}\right) \log_2 \left(n - \frac{2n}{\log_2 n}\right) - 4\left(n - \frac{2n}{\log_2 n}\right) \log_2 \log_2 n$$

$$
\begin{aligned}
&= n\log_2\left(n-\frac{2n}{\log_2 n}\right)-\frac{2n}{\log_2 n}\log_2\left(n-\frac{2n}{\log_2 n}\right)\\
&\quad -4n\log_2\log_2 n+\frac{8n}{\log_2 n}\log_2\log_2 n\\
&= n\log_2 n+n\log_2\left(1-\frac{2}{\log_2 n}\right)-2n-\frac{2n}{\log_2 n}\log_2\left(1-\frac{2}{\log_2 n}\right)\\
&\quad -4n\log_2\log_2 n+\frac{8n}{\log_2 n}\log_2\log_2 n\\
&\geq n\log_2 n-4n\log_2\log_2 n-2n\\
&\quad +n\log_2\left(1-\frac{2}{\log_2 n}\right)+\frac{8n}{\log_2 n}\log_2\log_2 n\\
&= n\log_2 n-4n\log_2\log_2 n-2n\\
&\quad +\frac{2n}{\log_2 n}\log_2\left(\left(1-\frac{2}{\log_2 n}\right)^{\frac{\log_2 n}{2}}(\log_2 n)^4\right)\\
&\geq n\log_2 n-4n\log_2\log_2 n-2n.
\end{aligned}
$$

The last inequality holds since $\left(1-\frac{2}{\log_2 n}\right)^{\frac{\log_2 n}{2}}$ is monotonically increasing in n and tends to $\frac{1}{e}$. This completes the proof. $\square$

Before showing the lower bound on $t(n,d)$ for each $d \geq 3$, we need the following theorem and obeservation.

Theorem 2. (Jarry and Laugier[7])

Any 2-edge-connected graph of order n and of odd diameter $p \geq 2$ contains at least $\left\lceil \frac{np-(2p+1)}{p-1} \right\rceil$ edges. Any 2-edge-connected graph of order n and of even diameter p contains at least $\min\left\{\left\lceil \frac{np-(2p+1)}{p-1} \right\rceil, \left\lceil \frac{(n-1)(p+1)}{p} \right\rceil\right\}$ *edges.*

Note that $\left\lceil \frac{np-(2p+1)}{p-1} \right\rceil = \left\lceil \frac{(n-1)(p-1)+n+p-1-(2p+1)}{p-1} \right\rceil = \left\lceil n-1+\frac{n-p-2}{p-1} \right\rceil \geq \left\lceil n-1+\frac{n-p-2}{p} \right\rceil = n-2+\left\lceil \frac{n-2}{p} \right\rceil$ and $\left\lceil \frac{(n-1)(p+1)}{p} \right\rceil = \left\lceil \frac{(n-1)p+n-1}{p} \right\rceil = \left\lceil n-1+\frac{n-1}{p} \right\rceil \geq \left\lceil n-1+\frac{n-p-2}{p} \right\rceil = n-2+\left\lceil \frac{n-2}{p} \right\rceil$. Thus, any 2-edge-connected graph of order n and of diameter $p \geq 2$ contains at least $n-2+\left\lceil \frac{n-2}{p} \right\rceil$ edges.

Let G be a graph and c be a rainbow connected coloring of G. It is easy to see that different bridges of G must receive different coloring under c. Therefore, the following observation is obvious.

Observation 1. *The rainbow connection number of a graph is at least the number of bridges in the graph.*

Proposition 2. *For* $3 \leq d < \lceil \frac{n}{2} \rceil$,

$$t(n,d) \geq n - d - 3 + \left\lceil \frac{n-1}{d} \right\rceil.$$

Proof. Let G be a k-rainbow connected graph of order n. Suppose that G has k bridges and G' is the graph obtained from G by deleting all the bridges. Then G' has $k+1$ components. We have $k \leq d$ by Observation 1. Suppose that $G_1, G_2, \cdots, G_{k_1}$ are the nontrivial components of G'. Thus, G' has $k_2 = k+1-k_1$ trivial components. Let n_i denote the order of G_i and d_i denote the diameter of G_i. We have that $n_1 + n_2 + \cdots + n_{k_1} = n - k_2$.

Claim 2. Each of the graph G_i is either a 2-edge-connected graph with diameter at least 2 or a complete graph of order at least 3.

Proof of Claim 2. Suppose that some of the graph G_i is neither a 2-edge-connected graph with diameter at least 2, nor a complete graph of order at least 3. That is, the graph G_i is a complete graph of order less than 3. Since G_i is nontrivial, G_i is a complete graph of order 2. However, the only edge of G_i is clearly a cut edge of G, a contradiction. Thus, this claim holds.

Now consider the number of edges of G_i. If G_i is a 2-edge-connected graph with diameter $d_i \geq 2$, then by Theorem 4 we have that $e(G_i) \geq n_i - 2 + \left\lceil \frac{n_i - 2}{d_i} \right\rceil$. If not, that is, G_i is a complete graph of order at least 3 by Claim 2. We have that $e(G_i) \geq \binom{n_i}{2} \geq n_i - 2 + \left\lceil \frac{n_i-2}{d_i} \right\rceil$. Thus, $e(G_i) \geq n_i - 2 + \left\lceil \frac{n_i-2}{d_i} \right\rceil$ for each i, where $1 \leq i \leq k_1$.

Claim 3. $d_i \leq d$.

Proof of Claim 3. Let x and y be two vertices in G_i. Since the shortest path connecting x and y in G must be a path contained G_i, we have $d_{G_i}(x,y) \leq d_G(x,y)$. Thus, $d_i \leq diam(G) \leq d$.

Now evaluate the number of edges in G. We have

$$\begin{aligned} e(G) &= k + \sum_{i=1}^{k_1} e(G_i) \\ &= k + \sum_{i=1}^{k_1} \left(n_i - 2 + \left\lceil \frac{n_i - 2}{d_i} \right\rceil \right) \end{aligned}$$

$$
\begin{aligned}
&\geq k+\sum_{i=1}^{k_1}\left(n_i-2+\left\lceil\frac{n_i-2}{d}\right\rceil\right)\\
&\geq k+\sum_{i=1}^{k_1}\left(n_i-2+\frac{n_i-2}{d}\right)\\
&= k+(n-k_2)-2k_1+\frac{(n-k_2)-2k_1}{d}\\
&= n-1-k_1+\frac{n-k-1-k_1}{d}\\
&\geq n-1-k+\frac{n-2k-1}{d}\\
&\geq n-d-1+\frac{n-2d-1}{d}\\
&= n-d-3+\frac{n-1}{d}.
\end{aligned}
$$

Thus, $e_d(n) \geq n-d-3+\lceil\frac{n-1}{d}\rceil$. □

Proposition 3. *For* $3 \leq d < \lceil\frac{n}{2}\rceil$, $t(n,d) \leq n-2+\lceil\frac{n}{d-1}\rceil$.

Proof. Set $n = q(d-1)+r$, where $0 < r \leq d-1$. Then $q = \lceil\frac{n}{d-1}\rceil - 1$. Pick q cycles of length d, say $C_1, C_2, \cdots C_q$. Identify u_i as a vertex u, where $u_i \in V(C_i)$ and $i = 1, 2, \ldots, q$. Finally, we attach r edges to the new vertex u. Denote by $G_d(n)$ the resulting graph, see Fig.1 for details. It is easy to check that $e(G_d(n)) = n-1+q = n-2+\lceil\frac{n}{d-1}\rceil$.

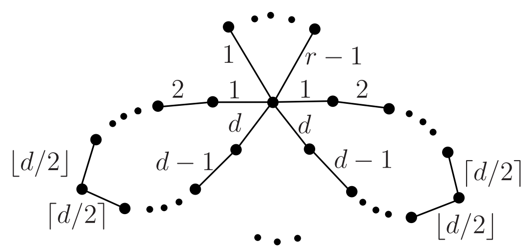

Fig.1. A d-rainbow connected coloring of the graph $G_d(n)$.

Now, we show that the graph $G_d(n)$ is d-rainbow connected. For each C_i, first, color an incident edge of u on C_i by 1, and the other one by d; second, color the edge adjacent to a 1-color edge on C_i by 2, and color the edge adjacent to a d-color edge on C_i by $d-1$. We do this until all edges of C_i are colored. For the r bridges,

we color them by different colors that have been used to color C_i. It is easy to check that the above coloring is a k-rainbow connected coloring. $\square$

Combining Propositions 1, 2 and 3, the following theorem holds.

Theorem 3. (i) *For sufficiently large* n, $t(n,2) \geq n\log_2 n - 4n\log_2\log_2 n - 2n$.

(ii) *For* $3 \leq d < \lceil \frac{n}{2} \rceil$, $n - d - 3 + \lceil \frac{n-1}{d} \rceil \leq t(n,d) \leq n - 2 + \lceil \frac{n}{d-1} \rceil$.

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM 244, Springer, 2008.

[2] M. Basavaraju, L.S. Chandran, D. Rajendraprasad, A. Ramaswamy, Rainbow connection number and radius, Graphs & Combin., to appear.

[3] S. Chakraborty, E. Fischer, A. Matsliah, R. Yuster, Hardness and algorithms for rainbow connection, J. Combin. Optimization 21(2011), 330–347.

[4] L.S. Chandran, A. Das, D. Rajendraprasad, N.M. Varma, Rainbow connection number and connected dominating sets, J. Graph Theory, to appear, also see Electronic Notes in Discrete Math. 38(2011), 239-244.

[5] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, Rainbow connection in graphs, Math. Bohem. 133(2008), 85-98.

[6] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, Rainbow connectivity of a graph, Networks 54(2009), 75-81.

[7] A. Jarry, A. Laugier, On the minimum number of edges of two-connected graphs with given diameter, Discrete Math. 312(2012), 757-762.

[8] M. Krivelevich, R. Yuster, The rainbow connection of a graph is (at most) reciprocal to its minimum degree, J. Graph Theory 63(2010), 185-91.

[9] H. Li, X. Li, S. Liu, Rainbow connection of graphs with diameter 2, Discrete Math. 312(2012), 1453-1457.

[10] X. Li, Y. Sun, Rainbow Connections of Graphs, SpringerBriefs in Math., Springer, New York, 2012.

[11] I. Schiermeyer, Rainbow connection in graphs with minimum degree three, Lecture Notes in Computer Science Vol.5874 (2009), 432-437.

[12] I. Schiermeyer, On minimally rainbow k-connected graphs, Discrete Appl. Math., doi:10.1016/j.dam.2011.05.001.